# ON AN APPLICATION OF HILBERT'S INDEPENDENCE THEOREM

## (OB ODNOM PRILOZHENII TEOREMY NEZAVISIMOSTI GIL'BERTA)

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**1.** Hilbert developed a method [1] by which, in addition to a criterion for the necessary and sufficient condition for the extremum of a functional in the calculus of variations, one can also obtain the fundamental conditions of the Hamilton-Jacobi theory.

Hilbert's method is based on the "independence theorem", by which the variational problem of minimizing some functional

$$J = \int_{(a)}^{(b)} F(x, y, y')\, dx = \text{extr} \qquad \left(y' = \frac{dy}{dx}\right) \tag{1.1}$$

can be reduced to a study of another function

$$J^* = \int_{(a)}^{(b)} (F(x, y, p) + (y' - p) F_p)\, dx \qquad \left(F_p = \frac{\partial F(x, y, p)}{\partial p}\right) \tag{1.2}$$

whose value no longer depends on the choice of a curve passing through the fixed end points $(a)$ and $(b)$.

Starting from the condition that the integral (1.2) is independent of the path of integration, it can be shown that the unknown function $p = p(x, y)$ of the variables $x$ and $y$ must satisfy the first-order partial differential equation

$$F_{pp}(p_x + pp_y) + pF_{py} + F_{px} - F_y = 0 \tag{1.3}$$

which Hilbert called the "adjoint equation" for the original problem (1.1). If the function $p(x, y)$ is chosen in this way, the problem of minimizing the functional (1.2) must be considered equivalent to problem (1.1).

Mayer [2] has studied the independence theorem and its relation to the Jacobi-Hamilton theorem for the case of $n$-dimensional space, when the desired functions minimizing the functional in question are required to satisfy a certain number (less than $n$) of differential equations.

**2.** We shall use Hilbert's independence theorem to find the trajectories of a point of unit mass ($m = 1$) moving in a conservative field with the potential $V(x, y)$. Consequently, by virtue of the principle of stationary action in the Jacobi form [3], we must here minimize a functional of the form (1.1), where

$$F(x, y, p) = \sqrt{2(E - V(x,y))}\sqrt{1 + p^2(x, y)} \qquad (2.1)$$

Constructing Hilbert's adjoint equation (1.3) for our variational problem, we obtain an equation which must be satisfied by the desired function $p(x, y)$

$$p_x + pp_y = (1 + p^2)(-p\Phi_x + \Phi_y) \qquad (\Phi = \ln\sqrt{2(E - V(x, y))}) \qquad (2.2)$$

Thus, the problem of finding trajectories for the motion of a point in a conservative field is now reduced to finding the characteristics of Hilbert's adjoint equation (2.2), or, expressing the problem differently, to the integration of the system

$$\frac{dx}{1} = \frac{dy}{p} = \frac{dp}{(1 + p^2)(-p\Phi_x + \Phi_y)} \qquad (2.3)$$

If we regard the upper limit in (1.1) as variable, then the value of the integral (1.1), taken along the extremal which passes through the points $(x_0, y_0)$ and $(x, y)$, will coincide with the value of the functional (1.2) for any curve passing through these two points $(x_0, y_0)$ and $(x, y)$, provided that $p(x, y)$ is a solution of Hilbert's adjoint equation (1.3). Hence, taking $F(x, y, p)$, in accordance with (2.1), we readily obtain

$$\frac{\partial J}{\partial x} = \frac{\sqrt{2(E - V)}}{\sqrt{1 + p^2}}, \qquad \frac{\partial J}{\partial y} = \frac{p\sqrt{2(E - V)}}{\sqrt{1 + p^2}}$$

Noting that, in accordance with (2.3), $p = dy/dx$, we can use the energy integral to obtain

$$\frac{\partial J}{\partial x} = v_x, \quad \frac{\partial J}{\partial y} = v_y \qquad (v = v_x + iv_y) \qquad (2.4)$$

and consequently we can write the equation of the trajectories in the following form:

$$\frac{\partial J}{\partial y}dx - \frac{\partial J}{\partial x}dy = 0 \qquad (2.5)$$

**3.** Let us consider a class of motions for which the function $F(x, y, y')$ is of the form

$$\sqrt{2(E - V(x, y))}\sqrt{1 + y'^2} = A(x, y) + B(x, y)y' \qquad (3.1)$$

and the integral (1.1) is independent of the shape of the curve joining the points $(x_0, y_0)$ and $(x, y)$; hence, by virtue of the condition that the integral is independent of the path of integration, we find [4]

$$\frac{\partial A(x, y)}{\partial y} - \frac{\partial B(x, y)}{\partial x} = 0$$

Since

$$A(x, y) = \partial J / \partial x, \qquad B(x, y) = \partial J / \partial y \qquad (3.2)$$

it follows that we shall satisfy the conditions of the class of motions under consideration if we require that the complex velocity of the point $\zeta = v \exp(-i\psi)$ (where $v$ is the magnitude of the velocity and $\psi$ is the angle formed by the velocity vector with the $x$-axis) be an analytic function of the complex variable $z = x + iy$. This in turn yields the condition, by (2.4), that

$$\zeta(z) = \frac{\partial J}{\partial x} - i\frac{\partial J}{\partial y} \qquad (\zeta = v\exp(-i\psi)) \qquad (3.3)$$

will also be an analytic function. Hence, using (2.5), we readily obtain the equation of the trajectories in this form

$$\mathrm{Im}\int_{z_0}^{z} \zeta(z)\,dz = 0 \qquad (3.4)$$

**4.** The necessary condition that must be satisfied by the potential of the force field $V(x, y)$ in order that the complex velocity $\zeta(z)$ be an analytic function can be obtained by setting

$$\triangle \ln v = 0 \qquad (v = \sqrt{2(E - V(x, y))}) \qquad (4.1)$$

Thus, the potential $V(x, y)$ must satisfy an equation of the form

$$V\triangle V = \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 \qquad (E = 0) \qquad (4.2)$$

This equation will be satisfied by any function of the form

$$V(x, y) = A \exp(\varphi(x, y)) \tag{4.3}$$

where $A$ is a constant and $\varphi(x, y)$ is a harmonic function.

It is known that if $f(z)$ is an analytic function in some region $G$, then $|f(z)|$ will be a logarithmically subharmonic function (that is, not only the function itself but also its logarithm will be subharmonic) in the same region, since it is known that the logarithm of the modulus of an analytic function, that is, an expression of the form $\ln|f(z)|$, is a subharmonic function [5].

It follows from this that if $A > 0$, then $V$ will belong to the class of logarithmically subharmonic functions, since the solution of (4.3) can always be represented in the following form:

$$V = A\,|\exp(f(z))| \qquad (\varphi(x, y) = \operatorname{Re} f(z))$$

where $f(z)$ is an analytic function.

It should be noted that if $A < 0$, then $V$ will be a superharmonic function, since for this case it will follow from (4.2) and (4.3) that $\Delta V < 0$. From the form of the solution (4.3), as well as from the properties of harmonic functions, it follows that if the functions $V_1$, $V_2$, ..., $V_n$ are solutions of the equation, then their product $V = V_1 \ldots V_n$ will also be a solution of equation (4.2); similarly, if $V_1$ and $V_2$ are solutions, then their quotient $V = V_1/V_2$ is also a solution; furthermore, if $V$ is a solution, then the function $f(V) = BV^{\alpha}$ is also a solution of equation (4.2) for arbitrary values of $B$ and $\alpha$.

In particular, if we consider a harmonic function of the form

$$\varphi = n \ln r \qquad (r = \sqrt{x^2 + y^2},\ n = \text{const})$$

then for the potential of the force field $V$ we obtain $V = Ar^n$, which represents attraction toward or repulsion from the center, depending on the signs of $A$ and $n$.

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